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Scaling Laws for the Multidimensional Burgers Equation with Quadratic External Potential

N.N. Leonenko[1] and M.D. Ruiz-Medina[2]

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The reordering of the multidimensional exponential quadratic operator in coordinate-momentum space (see X. Wang, C.H. Oh and L.C. Kwek (1998). *J. Phys. A.: Math. Gen.* **31**:4329–4336) is applied to derive an explicit formulation of the solution to the multidimensional heat equation with quadratic external potential and random initial conditions. The solution to the multidimensional Burgers equation with quadratic external potential under Gaussian strongly dependent scenarios is also obtained via the Hopf-Cole transformation. The limiting distributions of scaling solutions to the multidimensional heat and Burgers equations with quadratic external potential are then obtained under such scenarios.

KEY WORDS: nonhomogeneous multidimensional Burgers equation, quadratic external potential, scaling laws, spatiotemporal random fields, strongly dependent random initial conditions

AMS Subject Classifications: 60G60, 60G15, 62M15, 60H15

1. INTRODUCTION

Burgers equation arises in many physical problems such as cosmology, directed polymers in random media, vortex lines in superconductors, change density waves, etc. Hydrodynamics applications and the theory of turbulence constitute ones of the main motivations in the studies developed in the last few years. In both applied fields, external forces acting on the fluid are usually considered in the formulation of the Burgers equation. In the hydrodynamics setting the external forces

[1] Cardiff School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF24 4AG, United Kingdom.
[2] Department of Statistics and Operations Research University of Granada, Campus de Fuente Nueva s/n, E-18071 Granada, Spain.

are assumed to be smooth (large-scale forcing). However, in its mathematical representation, periodic functions are usually considered, referring the mathematical treatment to the compact spatial situation. This strong assumption has important consequences in the physical modelling, and it is not always realistic. In this paper, we study the Burgers equation with quadratic external potentials on $\mathbb{R}^n$. This case is very important from the physical point of view (e.g. hydrodynamics setting) since allows to go beyond the compact spatial case. The importance of quadratic potential can be seen also from the role of the so-called oscillator (or oscillator process) in the stochastic spectral theory for self-adjoint operators (see Simon,[1] p. 36 or Demuth and van Casteren,[2] p. 22).

Different versions of Burgers equation (and heat equation) with random initial potential have been extensively studied since its initial formulation in Rosenblatt[3,4] and Sinai,[5] considering Gaussian and non-Gaussian scenarios (see Refs. 6–18 among others). Probabilistic and statistical properties of Burgers equation with external random forcing are also analyzed, for example, in Sinai,[5] Hoang and Khanin;[19] Holden *et al.*,[20] Molchanov *et al.*,[21] and Saichev and Woyczynski.[22] Recent books of Woyczynski[23] and Leonenko[24] contain an extensive bibliography on the subject and expositions of some of the main results of the theory of Burgers turbulence. In this paper, we consider deterministic large-scale external forcing and Gaussian random initial potential in the asymptotic study of Burgers equation.

Specifically, we address the problem of defining the limiting distribution of a suitable scaling of the solution of the heat and Burgers equations with quadratic external potentials, under Gaussian strongly dependent initial velocity potential. Note that the normalization procedures used here differs considerably from the ones considered without external potentials. New multidimensional formulae, which describe explicit solutions of the heat and Burgers equations with quadratic external potential, are derived, providing a good potential for other approaches in Burgers turbulence problem. The paper generalizes the results given in Barndorff-Nielsen and Leonenko,[25] in relation to formulation of a solution for the heat equation on $\mathbb{R}^n$ and the multidimensional Burgers equation with quadratic external potentials.

Let $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, be the vector random field solution to the multidimensional Burgers equation problem

$$\begin{aligned} &\frac{\partial}{\partial t}\mathbf{u} + \langle \mathbf{u}, \nabla\rangle \mathbf{u} = \mu\Delta\mathbf{u} + 2\mu\nabla\Phi \\ &\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) = \nabla U(\mathbf{x}),\ \ \mathbf{x} \in \mathbb{R}^n\ , \end{aligned} \tag{1}$$

where the scalar field U represents the initial velocity potential. Here, as usual, ∇ denotes the gradient and Δ denotes the Laplacian. We consider the case of quadratic external potentials, that is, the case where

$$\Phi = a + b\|\mathbf{x}\|^2 = a + b\left(\sum_{i=1}^{n} x_i^2\right), \ b > 0.$$

The solution **u** to problem (1) can be defined as the Hopf-Cole transformation

$$\mathbf{u}(t, \mathbf{x}) = -2\mu\nabla \ln h(t, \mathbf{x}) \tag{2}$$

of the solution h to the following heat equation:

$$\frac{\partial}{\partial t}h = \mu\Delta h - \Phi h$$
$$h(0, \mathbf{x}) = h_0(x) = G(u(\mathbf{x})), \ \mathbf{x} \in \mathbb{R}^n \tag{3}$$

(see, for example Refs. 5, 21, 26).

The initial velocity potential U is assumed to be given by an homogeneous and isotropic random field ξ with spectral density

$$f_\xi(\lambda) = \frac{(2\pi)^\gamma}{\varphi(\gamma)}\|\lambda\|^{-n+2\gamma}(1 + \|\lambda\|^2)^{-\alpha}, \ \gamma > 0, \alpha - \gamma > 0, \tag{4}$$

where

$$\varphi(\gamma) = \frac{\pi^{n/2}2^\gamma\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{n}{2} - \frac{\gamma}{2}\right)}.$$

Hence, ξ admits a spectral representation in the L_2-stochastic integral form given by:

$$\xi(\mathbf{x}) = \int_{\mathbb{R}} \exp\{i\langle\lambda, \mathbf{x}\rangle\}Z_\xi(d\lambda) = \int_{\mathbb{R}^n} \exp\{i\langle\lambda, \mathbf{x}\rangle\}f_\xi^{1/2}(\lambda)W(d\lambda), \tag{5}$$

with the associated spectral representation of its covariance function

$$B_\xi(\mathbf{x}) = \mathrm{cov}(\xi(0), \xi(\mathbf{x})) = \int_{\mathbb{R}^n} \exp\{i\langle\lambda, \mathbf{x}\rangle\}F_\xi(d\lambda),$$

in terms of the bounded and positive spectral measure F_ξ, which defines the second-order structure of the complex-valued orthogonally scattered random measure $Z_\xi(\cdot)$ such that $E|Z_\xi(d\lambda)|^2 = F_\xi(d\lambda)$, and with

$$F_\xi(d\lambda) = f_\xi(\lambda)d\lambda.$$

Here, W represents Gaussian white noise. For $\gamma < n/2$, this model displays long-range dependence (see, for example Ref. 27).

The (non-linear) function

$$G(u) = \exp\left\{-\frac{u}{2\mu}\right\}, \quad u \in \mathbb{R}, \tag{6}$$

admits a series expansion in the Hilbert space $(L_2(\mathbb{R}^n), \varphi(u)du)$ in terms of orthogonal Chebyshev-Hermite polynomials given by

$$G(u) = \sum_{k=0}^{\infty} \frac{C_k H_k(u)}{k!}, \quad C_k = \int_{\mathbb{R}^n} G(u) H_k(u) \varphi(u) du,$$

where

$$H_k(u) = (-1)^k [\varphi(u)]^{-1} \frac{d^k}{du^k} \varphi(u), \quad k = 0, 1, 2, \ldots,$$

with

$$\varphi(u) = \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{u^2}{2}\right\}, \quad u \in \mathbb{R}^n.$$

Thus,

$$h_0(\mathbf{x}) = \exp\left\{-\frac{\xi(\mathbf{x})}{2\mu}\right\} = \sum_{k=0}^{\infty} \frac{C_k}{k!} H_k(\xi(\mathbf{x})),$$

where

$$S(\mathbf{x}) = \operatorname{cov}(h_0(\mathbf{0}), h_0(\mathbf{x})) = B_{h_0}(\mathbf{x}) = \sum_{k=1}^{\infty} \frac{C_k^2}{k!} B_{\xi}^k(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n. \tag{7}$$

Note that

$$C_0 = \exp\left\{\frac{1}{8\mu^2}\right\}, C_1 = -\frac{1}{2\mu} \exp\left\{\frac{1}{8\mu^2}\right\}.$$

The stationary process $h_0(\mathbf{x}) = \exp\left\{-\frac{\xi(\mathbf{X})}{2\mu}\right\}$, $\mathbf{x} \in \mathbb{R}^n$, then admits the spectral representation in the form of L_2-stochastic integral:

$$h_0(\mathbf{x}) = \exp\left\{-\frac{\xi(\mathbf{x})}{2\mu}\right\} = \exp\left\{\frac{1}{8\mu^2}\right\} + \int_{\mathbb{R}^n} \exp\{i\langle\lambda, \mathbf{x}\rangle\} Z_{h_0}(d\lambda), \tag{8}$$

with the associated spectral representation of its covariance function

$$B_{h_0}(\mathbf{x}) = \operatorname{cov}(h_0(\mathbf{0}), h_0(\mathbf{x})) = \int_{\mathbb{R}^n} \exp\{i\langle\lambda, \mathbf{x}\rangle\} F_{h_0}(d\lambda),$$

in terms of the bounded and positive spectral measure F_{h_0}, which defines the second-order structure of the complex-valued orthogonally scattered random measure $Z_{h_0}(\cdot)$ such that $E|Z_{h_0}(d\lambda)|^2 = F_{h_0}(d\lambda)$.

2. SCALING LAW FOR THE HEAT EQUATION ON $\mathbb{R}^n$ WITH QUADRATIC EXTERNAL POTENTIAL

In this section, an explicit solution to problem (3) is derived in Lemma 1. A suitable scaling of such a solution is considered to obtain the convergence of the finite-dimensional distributions of its generalized version on a fractional Sobolev space. The order of such a space involves the long-memory parameter of the initial velocity potential (see Theorem 1).

We have considered with $\omega = 2\sqrt{\mu b}$ and $\varepsilon \to 0$ the rescaling procedure

$$t \mapsto \frac{1}{\omega}\ln\frac{t}{\varepsilon};\ x \mapsto \frac{xt}{\varepsilon^2};\ 0 < \varepsilon < t.$$

Our rescaling procedures differ significantly from the rescaling procedure

$$t \mapsto \frac{t}{\varepsilon};\ x \mapsto \frac{x}{\sqrt{\varepsilon}};$$

of the heat equation and Burgers equation with external potential $\Phi = 0$ (see Albeverio *et al.*,[6] Leonenko and Woyczynski,[(11,12)] Woyczynski,[(23)] Leonenko,[(24)] Anh and Leonenko,[(15–17)] Ruiz-Medina, Angulo and Anh[(13)] and references therein).

Lemma 1. *The solution to problem (3) is given by*

$$\begin{aligned} h(t,\boldsymbol{x}) &= \frac{\exp\{-at\}}{\left(\cosh(\omega t)2\pi\left(\frac{\mu}{b}\right)^{1/2}\tanh(\omega t)\right)^{n/2}}\exp\left\{-1/2\left(\frac{b}{\mu}\right)^{1/2}\tanh(\omega t)\|\mathbf{x}^2\|\right\}\\ &\quad\times\int_{\mathbb{R}^n}\exp\left\{-\frac{\|\frac{\mathbf{x}}{\cosh(\omega t)}-\mathbf{x}'\|^2}{2(\frac{\mu}{b})^{1/2}\tanh(\omega t)}\right\}h(0,\mathbf{x}')d\mathbf{x}'\\ &= \frac{\exp\{-at\}}{(\cosh(\omega t))^{n/2}}\exp\left\{-1/2\left(\frac{b}{\mu}\right)^{1/2}\tanh(\omega t)\|\mathbf{x}\|^2\right\}\\ &\quad\times\int_{\mathbb{R}^n}\exp\left\{i\left\langle\lambda,\frac{\mathbf{x}}{\cosh(\omega t)}\right\rangle\right\}\exp\left\{-\left(\frac{\mu}{b}\right)^{1/2}\frac{\tanh(\omega t)}{2}\|\lambda\|^2\right\}Z_{ho}(d\lambda), \end{aligned} \tag{9}$$

where Z is defined as in equation (8), $\|\lambda\|^2 = \sum_{i=1}^{n}\lambda_i^2$, *and*

$$\omega = 2(\mu b)^{1/2}.$$

Remark 1. Note that the paper by Wang[(28)] (containing the reordering of multidimensional exponential quadratic operators in coordinate-momentum space) is useful but do not contain sufficient information about solution of Burgers equation with quadratic potential in multidimansional case.

Proof: Formally, the solution h to problem (3) is given by

$$h(t, \mathbf{x}) = T_t h_0 = \exp\{t(\mu\Delta - a - b\|\mathbf{x}\|^2)\}h_0,$$

in terms of the semigroup $T_t = \exp\{t(\mu\Delta - a - b\|\mathbf{x}\|^2)\}$. Function h then belongs to the space $\mathcal{B}[[0,\infty); H^1(\mathbb{R}^n)]$ of bounded continuous functions on time with values in the Sobolev space $H^1(\mathbb{R}^n)$. Note that the quadratic external potential modifies the spatial decay properties of the solution over time, lessening the decay velocity.

The semigroup

$$\tilde{T}_t = \exp\{t(\mu\Delta - b\|\mathbf{x}\|^2)\}$$

can be rewritten as an exponential quadratic operator,

$$\tilde{T}_t = \exp\left\{\frac{1}{2}\mathbf{Y}^*\mathbf{B}_t\mathbf{Y}\right\}, \tag{10}$$

where $\mathbf{Y}^* = (x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$, and

$$\mathbf{B}_t = \left[\begin{array}{c|c} -2tbI_{n\times n} & 0 \\ \hline 0 & 2t\mu I_{n\times n}\end{array}\right].$$

We then apply the results from Wang *et al.*[28] to reordering the multidimensional exponential quadratic operator in coordinate-momentum space. Specifically, from the identity

$$\begin{aligned}
&\exp\left\{\left[\begin{array}{c|c} -2tbI_{n\times n} & 0 \\ \hline 0 & 2t\mu I_{n\times n}\end{array}\right]\left[\begin{array}{c|c} 0 & I_{n\times n} \\ \hline -I_{n\times n} & 0\end{array}\right]^{-1}\right\} \\
&= \left[\begin{array}{c|c} \cosh(\omega t I_{n\times n}) & 2tb(I_{n\times n})\sinh(\omega t I_{n\times n})(\omega t I_{n\times n})^{-1} \\ \hline 2t\mu I_{n\times n}\sinh(\omega t I_{n\times n})(\omega t I_{n\times n})^{-1} & \cosh(\omega t I_{n\times n})\end{array}\right] \\
&= \left[\begin{array}{c|c} \cosh(\omega t I_{n\times n}) & \left(\frac{b}{\mu}\right)^{1/2} I_{n\times n}\sinh(\omega t I_{n\times n}) \\ \hline \left(\frac{b}{\mu}\right)^{1/2} I_{n\times n}\sinh(\omega t I_{n\times n}) & \cosh(\omega t I_{n\times n}),\end{array}\right],
\end{aligned} \tag{11}$$

the following expression is obtained for the exponential quadratic operator defining $\tilde{T}_t$:

$$\begin{aligned}
&\exp\left\{\frac{1}{2}\mathbf{Y}^*\mathbf{B}_t\mathbf{Y}\right\} \\
&\quad = \exp\left\{\frac{1}{2}\mathrm{tr}\,\mathbf{V}\right\}\exp\left\{-\frac{1}{2}\mathbf{x}\mathbf{W}\mathbf{x}^T\right\}\exp\{\mathbf{x}\mathbf{V}\nabla\}\exp\left\{\frac{1}{2}\nabla\mathbf{Z}\nabla^T\right\},
\end{aligned} \tag{12}$$

where

$$\mathbf{W} = \left(\frac{b}{\mu}\right)^{1/2} (I_{n\times n}) \sinh(\omega t I_{n\times n})[\cosh(\omega t I_{n\times n})]^{-1}$$

$$\mathbf{Z} = [\cosh(\omega t I_{n\times n})]^{-1} \left(\frac{\mu}{b}\right)^{1/2} I_{n\times n} \sinh(\omega t I_{n\times n})$$

$$\mathbf{V} = -\ln[\cosh(\omega t I_{n\times n})], \tag{13}$$

with $\omega = 2(\mu b)^{1/2}$. Therefore,

$$\begin{aligned} h(t,\mathbf{x}) &= \exp\{-at\}[\cosh(\omega t)]^{-n/2} \exp\left\{-\frac{1}{2}\left(\frac{b}{\mu}\right)^{1/2} \tanh(\omega t)\|\mathbf{x}\|^2\right\} \\ &\quad \times \exp\{-\ln[\cosh(\omega t)]\langle \mathbf{x}, \nabla\rangle\} \exp\left\{\frac{1}{2}\left(\frac{\mu}{b}\right)^{1/2} \tanh(\omega t)\Delta\right\} h_0(\mathbf{x}) \\ &= \exp\{-at\}[\cosh(\omega t)]^{-n/2} \exp\left\{-\frac{1}{2}\left(\frac{b}{\mu}\right)^{1/2} \tanh(\omega t)\|\mathbf{x}\|^2\right\} \\ &\quad \times \frac{1}{\left(2\pi\left(\frac{\mu}{b}\right)^{1/2}\tanh(\omega t)\right)^{n/2}} \int_{\mathbb{R}^n} \exp\left\{-\frac{\|\frac{\mathbf{x}}{\cosh(\omega t)} - \mathbf{x}'\|^2}{2\left(\frac{\mu}{b}\right)^{1/2}\tanh(\omega t)}\right\} h(0,\mathbf{x}')d\mathbf{x}'. \end{aligned} \tag{14}$$

That is, equation (9) holds. □

Theprem 1. Let h be the solution to problem (3) defined in Lemma 1. Then, the finite-dimensional distributions of the generalized random field $\widetilde{\mathcal{H}}_\varepsilon$ defined by the ordinary random field

$$H_\varepsilon(t,\mathbf{x}) = \frac{\exp\left\{\frac{1}{2}\left(\frac{b}{\mu}\right)^{1/2} \|\frac{\mathbf{x}t}{\varepsilon^2}\|^2 \left[\frac{\varepsilon^2-t^2}{\varepsilon^2+t^2}\right]\right\}}{\varepsilon^{a/\omega+n/2+\gamma}} h\left(\frac{1}{\omega}\ln\left(\frac{t}{\varepsilon}\right), \frac{\mathbf{x}t}{\varepsilon^2}\right),$$

$$\omega = 2(\mu b)^{1/2},\ 0<\varepsilon<t, \tag{15}$$

converge, when ε goes to zero, to the finite-dimensional distributions of the Gaussian generalized random field $\widetilde{\mathcal{H}}$ with zero mean and covariance function

$$\begin{aligned} E\left[\widetilde{H}(t_1,\phi)\widetilde{H}(t_2,\varphi)\right] &= R(t_1,\phi,t_2,\varphi) \\ &= (t_1t_2)^{-(a/\omega+n/2)} 2^{1-2\gamma} \int_{\mathbb{R}}\int_{\mathbb{R}^n} \phi(\mathbf{x}_1)\|\mathbf{x}_1-\mathbf{x}_2\|^{-2\gamma}\varphi(\mathbf{x}_2)d\mathbf{x}_1 d\mathbf{x}_2 \end{aligned} \tag{16}$$

for $0<\gamma<n/2$, *and* $\phi, \varphi \in H^{\gamma-n/2}(\mathbb{R}^n)$.

Proof: From (9),

$$E\left[h\left(\frac{1}{\omega}\ln\left[\frac{t_1}{\varepsilon}\right],\frac{\mathbf{x}_1 t_1}{\varepsilon^2}\right)h\left(\frac{1}{\omega}\ln\left[\frac{t_2}{\varepsilon^2}\right],\frac{\mathbf{x}_2 t_2}{\varepsilon^2}\right)\right]$$

$$= A\left(\frac{1}{\omega}\ln\left[\frac{t_1}{\varepsilon}\right],\frac{1}{\omega}\ln\left[\frac{t_2}{\varepsilon}\right],\frac{\mathbf{x}_1 t_1}{\varepsilon^2},\frac{\mathbf{x}_2 t_2}{\varepsilon^2}\right)$$

$$\times\int_{\mathbb{R}^n}\exp\left\{i\left\langle\frac{\lambda}{\varepsilon},\left[\frac{2\mathbf{x}_1 t_1^2}{t_1^2+\varepsilon^2}-\frac{2\mathbf{x}_2 t_2^2}{t_2^2+\varepsilon^2}\right]\right\rangle\right\}$$

$$\times\exp\left\{-\frac{\|\lambda\|^2}{2}\left(\frac{\mu}{b}\right)^{1/2}\left[\frac{t_1^2-\varepsilon^2}{t_1^2+\varepsilon^2}+\frac{t_2^2-\varepsilon^2}{t_2^2+\varepsilon^2}\right]\right\}$$

$$\times\left[\int_{\mathbb{R}^n}\exp\{-i\langle\lambda,\mathbf{x}\rangle\}\sum_{k=1}^{\infty}\frac{C_k^2}{k!}B_\xi^k(\mathbf{x})d\mathbf{x}\right]d\lambda, \tag{17}$$

where

$$A\left(\frac{1}{\omega}\ln\left[\frac{t_1}{\varepsilon}\right],\frac{1}{\omega}\ln\left[\frac{t_2}{\varepsilon}\right],\frac{\mathbf{x}_1 t_1}{\varepsilon^2},\frac{\mathbf{x}_2 t_2}{\varepsilon^2}\right)$$

$$=\frac{2^n\left(\frac{t_1t_2}{\varepsilon^2}\right)^{-a/\omega}(t_1t_2)^{n/2}\varepsilon^n\exp\left\{-\frac{1}{2}\left(\frac{b}{\mu}\right)^{1/2}\left[\|\frac{x_1t_1}{\varepsilon^2}\|^2\frac{t_1^2-\varepsilon^2}{t_1^2+\varepsilon^2}+\|\frac{x_2t_2}{\varepsilon^2}\|^2\frac{t_2^2-\varepsilon^2}{t_2^2+\varepsilon^2}\right]\right\}}{[(t_1t_2)^2+(t_2\varepsilon)^2+(t_1\varepsilon)^2+\varepsilon^4]^{n/2}}.$$

When ε goes to zero, the limit of

$$\frac{\exp\left\{\frac{1}{2}\left(\frac{b}{\mu}\right)^{1/2}\|\frac{\mathbf{x}t}{\varepsilon^2}\|^2\left[\frac{\varepsilon^2-t^2}{\varepsilon^2+t^2}\right]\right\}}{\varepsilon^{a/\omega+n/2}}A\left(\frac{1}{\omega}\ln\left[\frac{t_1}{\varepsilon}\right],\frac{1}{\omega}\ln\left[\frac{t_2}{\varepsilon}\right],\frac{\mathbf{x}_1 t_1}{\varepsilon^2},\frac{\mathbf{x}_2 t_2}{\varepsilon^2}\right)$$

is then given by

$$\frac{2^n(t_1t_2)^{-a/\omega+n/2}}{(t_1t_2)^n}. \tag{18}$$

Considering the change of variable $\widetilde{\lambda}=\frac{\lambda}{\varepsilon}$,

$$\int_{\mathbb{R}^n}\exp\left\{i\left\langle\frac{\lambda}{\varepsilon},\left[\frac{2\mathbf{x}_1 t_1^2}{t_1^2+\varepsilon^2}-\frac{2\mathbf{x}_2 t_2^2}{t_2^2+\varepsilon^2}\right]\right\rangle\right\}\exp\left\{-\frac{\|\lambda\|^2}{2}\left(\frac{\mu}{b}\right)^{1/2}\left[\frac{t_1^2-\varepsilon^2}{t_1^2+\varepsilon^2}+\frac{t_2^2-\varepsilon^2}{t_2^2+\varepsilon^2}\right]\right\}$$

$$\times\left[\int_{\mathbb{R}^n}\exp\{-i\langle\lambda,\mathbf{x}\rangle\}\sum_{k=1}^{\infty}\frac{C_k^2}{k!}B_\xi^k(\mathbf{x})d\mathbf{x}\right]d\lambda$$

$$= \int_{\mathbb{R}^n} \exp\left\{ i \left\langle \widetilde{\lambda}, \left[\frac{2\mathbf{x}_1 t_1^2}{t_1^2+\varepsilon^2} - \frac{2\mathbf{x}_2 t_2^2}{t_2^2+\varepsilon^2} \right] \right\rangle \right\} \exp\left\{ -\frac{\|\widetilde{\lambda}\varepsilon\|^2}{2} \left(\frac{\mu}{b}\right)^{1/2} \left[\frac{t_1^2-\varepsilon^2}{t_1^2+\varepsilon^2} + \frac{t_2^2-\varepsilon^2}{t_2^2+\varepsilon^2} \right] \right\}$$

$$\times \left[\int_{\mathbb{R}^n} \exp\left\{ -i \left\langle \widetilde{\lambda}\varepsilon, \mathbf{x} \right\rangle \right\} \sum_{k=1}^{\infty} \frac{C_k^2}{k!} B_\xi^k(\mathbf{x}) d\mathbf{x} \right] \varepsilon^n d\widetilde{\lambda}.$$

$$= \int_{\mathbb{R}^n} \exp\left\{ i \left\langle \widetilde{\lambda}, \left[\frac{2\mathbf{x}_1 t_1^2}{t_1^2+\varepsilon^2} - \frac{2\mathbf{x}_2 t_2^2}{t_2^2+\varepsilon^2} \right] \right\rangle \right\} \exp\left\{ -\frac{\|\widetilde{\lambda}\varepsilon\|^2}{2} \left(\frac{\mu}{b}\right)^{1/2} \left[\frac{t_1^2-\varepsilon^2}{t_1^2+\varepsilon^2} + \frac{t_2^2-\varepsilon^2}{t_2^2+\varepsilon^2} \right] \right\}$$

$$\times \left[\int_{\mathbb{R}^n} \exp\left\{ -i \left\langle \widetilde{\lambda}\varepsilon, \mathbf{x} \right\rangle \right\} \sum_{k=2}^{\infty} \frac{C_k^2}{k!} B_\xi^k(\mathbf{x}) d\mathbf{x} \right] \varepsilon^n d\widetilde{\lambda}$$

$$+ \int_{\mathbb{R}^n} \exp\left\{ i \left\langle \widetilde{\lambda}, \left[\frac{2\mathbf{x}_1 t_1^2}{t_1^2+\varepsilon^2} - \frac{2\mathbf{x}_2 t_2^2}{t_2^2+\varepsilon^2} \right] \right\rangle \right\} \exp\left\{ -\frac{\|\widetilde{\lambda}\varepsilon\|^2}{2} \left(\frac{\mu}{b}\right)^{1/2} \left[\frac{t_1^2-\varepsilon^2}{t_1^2+\varepsilon^2} + \frac{t_2^2-\varepsilon^2}{t_2^2+\varepsilon^2} \right] \right\}$$

$$\times \frac{(2\pi)^\gamma}{\varphi(\gamma)} \|\widetilde{\lambda}\varepsilon\|^{-n+2\gamma} (1 + \|\widetilde{\lambda}\varepsilon\|^2)^{-\alpha} \varepsilon^n d\widetilde{\lambda}$$

$$= I_1\left(\frac{1}{\omega} \ln\left[\frac{t}{\varepsilon}\right], \frac{\mathbf{x}t}{\varepsilon^2} \right) + I_2\left(\frac{1}{\omega} \ln\left[\frac{t}{\varepsilon}\right], \frac{\mathbf{x}t}{\varepsilon^2} \right).$$

Since, for ε sufficiently small,

$$\varepsilon^n \left[\int_{\mathbb{R}} \exp\left\{ -i \left\langle \varepsilon\widetilde{\lambda}, \mathbf{x} \right\rangle \right\} \sum_{k=2}^{\infty} \frac{C_k^2}{k!} B_\xi^k(\mathbf{x}) d\mathbf{x} \right]$$

$$\leq \varepsilon^n \left[\int_{\mathbb{R}} \exp\left\{ -i \left\langle \varepsilon\widetilde{\lambda}, \mathbf{x} \right\rangle \right\} B_\xi^2(\mathbf{x}) d\mathbf{x} \right] \sum_{k=2}^{\infty} \frac{C_k^2}{k!},$$

and

$$\varepsilon^n \int_{\mathbb{R}^n} \exp\left\{ -i \left\langle \varepsilon\widetilde{\lambda}, \mathbf{x} \right\rangle \right\} B_\xi^2(\mathbf{x}) d\mathbf{x}$$

goes to zero like function $\varepsilon^{4\gamma}$ (see Ruiz-Medina, Angulo and Anh),(13) then,

$$\frac{I_1\left(\frac{1}{\omega} \ln\left[\frac{t}{\varepsilon}\right], \frac{\mathbf{x}t}{\varepsilon^2} \right)}{\varepsilon^{2\gamma}} \longrightarrow 0 \tag{19}$$

when $\varepsilon \to 0$.

Moreover,

$$\frac{I_2\left(\frac{1}{\omega} \ln\left[\frac{t}{\varepsilon}\right], \frac{\mathbf{x}t}{\varepsilon^2} \right)}{\varepsilon^{2\gamma}} = \int_{\mathbb{R}^n} \exp\left\{ i \left\langle \widetilde{\lambda}, \left[\frac{2\mathbf{x}_1 t_1^2}{t_1^2+\varepsilon^2} - \frac{2\mathbf{x}_2 t_2^2}{t_2^2+\varepsilon^2} \right] \right\rangle \right\} \exp\left\{ -\frac{\|\widetilde{\lambda}\varepsilon\|^2}{2} \left(\frac{\mu}{b}\right)^{1/2} \right.$$

$$\left. \times \left[\frac{t_1^2-\varepsilon^2}{t_1^2+\varepsilon^2} + \frac{t_2^2-\varepsilon^2}{t_2^2+\varepsilon^2} \right] \right\} \frac{(2\pi)^\gamma}{\varphi(\gamma)} \|\widetilde{\lambda}\|^{-n+2\gamma} (1 + \|\widetilde{\lambda}\varepsilon\|^2)^{-\alpha} d\widetilde{\lambda}, \tag{20}$$

which when $\varepsilon \to 0$ tends to

$$\frac{(2\pi)^{\gamma}}{\varphi(\gamma)} \int_{\mathbb{R}^n} \exp\{i\langle\widetilde{\lambda}, 2(\mathbf{x}_1 - \mathbf{x}_2)\rangle\}\|\widetilde{\lambda}\|^{-n+2\gamma} d\widetilde{\lambda} = 2^{-2\gamma}\|\mathbf{x}_1 - \mathbf{x}_2\|^{-2\gamma}. \tag{21}$$

Here, we have used that the function $\|\widetilde{\lambda}\|^{-n+2\gamma}$ defines the kernel, in the Schwartz sense, of the Riesz operator of order $2\gamma(0 < \gamma < n/2)$

$$\begin{aligned}(-\Delta)_{\lambda}^{-\gamma} \hat{f}(\lambda) &\underset{\text{w.s.}}{=} \int_{\mathbb{R}^n} \|\lambda - \omega\|^{-n+2\gamma} \hat{f}(\omega) d\omega \\ &\underset{\text{w.s.}}{=} \int_{\mathbb{R}^n} \exp\{-\langle i\lambda, \mathbf{x}\rangle\}\|x\|^{-2\gamma} f(x) dx \end{aligned} \tag{22}$$

(see, for example Refs. 29, 30), with w.s. denoting the identity in the weak-sense, that is, in the sense of tempered distributions.

From equations (5), (18), (19) and (20)–(21), the convergence, when $\varepsilon \to 0$, of the finite-dimensional distributions of the generalized random field $\widetilde{\mathcal{H}}_{\varepsilon}$ to the finite-dimensional distributions of the Gaussian generalized random field $\widetilde{\mathcal{H}}$ follows. □

2. SCALING LAW FOR THE MULTIDIMENSIONAL BURGERS EQUATION

In this section, we apply a suitable scaling in space to obtain the limiting distribution of the solution to the Burgers equation with initial velocity potential defined from equation (4)–(5). The limiting Gaussian generalized random field obtained with such a scaling has long-range dependence for $\gamma + 1/n < n/2$, and defines, in the weak-sense, an ordinary fractal random field with infinite variance. For $\gamma + 1/n \geq n/2$, the limiting Gaussian random field is an intrinsic random field which must be fractionally integrated to obtain its pointwise ordinary definition. The fractional Sobolev space $H^{\gamma - n/2 + 1/n}(\mathbb{R}^n)$ of order $\gamma - n/2 + 1/n$ is considered in the definition of the limiting Gaussian generalized random field, since the functions in this space satisfy suitable local regularity and moments conditions according to the local singularity and asymptotic behaviour of the functions in the associated reproducing kernel Hilbert space (see Refs. 23, 31).

Let $\mathbf{u}(t, \mathbf{x}) = (u_1(t, \mathbf{x}), \ldots, u_n(t, \mathbf{x})), t > 0, \mathbf{x} \in \mathbb{R}^n$, be the vector velocity random field given by

$$\mathbf{u}(t, \mathbf{x}) = -2\mu\nabla \ln(h(t, \mathbf{x})) = -2\mu\nabla\left[-\frac{1}{2}\mathbf{x}\mathbf{W}\mathbf{x}^T\right]$$
$$-2\mu \frac{\nabla\left[\exp\{\mathbf{x}\mathbf{V}\nabla\}\exp\left\{\frac{1}{2}\nabla\mathbf{Z}\nabla^T\right\}h_0\right](t, \mathbf{x})}{\left[\exp\{\mathbf{x}\mathbf{V}\nabla\}\exp\left\{\frac{1}{2}\nabla\mathbf{z}\nabla^T\right\}h_0\right](t, \mathbf{x})}$$

$$= \sqrt{\mu b}\tanh(\omega t)\nabla\|\mathbf{x}\|^2 - 2\mu \frac{\nabla\left[\exp\left\{\frac{1}{2}\nabla\mathbf{Z}\nabla^T\right\} h_0\left(t, \frac{\mathbf{x}}{\cosh(\omega t)}\right)\right]}{\left[\exp\left\{\frac{1}{2}\nabla\mathbf{Z}\nabla^T\right\} h_0\right]\left(t, \frac{\mathbf{x}}{\cosh(\omega t)}\right)}. \quad (23)$$

In particular, we have

$$u_j(t,\mathbf{x}) = 2\sqrt{\mu b}x_j \tanh(\omega t) + 4\mu \frac{\int_{\mathbb{R}^n} \frac{x_j - y_j\cosh(\omega t)}{\sqrt{\mu/b}\sinh(2\omega t)} \exp\left\{-\frac{[x-y\cosh(\omega t)]^2}{\sqrt{\mu/b}\sinh(2\omega t)} - \frac{U(y)}{2\mu}\right\} d\mathbf{y}}{\int_{\mathbb{R}^n} \exp\left\{-\frac{[x-\mathbf{y}\cosh(\omega t)]^2}{\sqrt{\mu/b}\sinh(2\omega t)} - \frac{U(\mathbf{y})}{2\mu}\right\} d\mathbf{y}}, \; j = 1,\ldots,n. \quad (24)$$

Theprem 2. *Let* $\boldsymbol{u}$ *be the solution (23)–(24) to the Burgers equation (1) with quadratic external potential, considering random initial velocity potential* $U = \xi$ *with spectral density (4), and with* G *being defined as in (6). Then, the finite-dimensional distributions of the vector generalized random field* $\mathcal{U}^\varepsilon$ *associated with the ordinary random field*

$$\mathbf{U}^\varepsilon(t,\mathbf{x}) = \frac{1}{\varepsilon^{n/2+2+\gamma}}\left(\frac{(2\pi)^\gamma}{\varphi(\gamma)}\right)^{-1/2} \times \left[\mathbf{u}\left(\frac{1}{\omega}\ln\left[\frac{t}{\varepsilon}\right], \frac{\mathbf{x}t}{\varepsilon^2}\right) - \mathbf{F}\left(\frac{1}{\omega}\ln\left[\frac{t}{\varepsilon}\right], \frac{\mathbf{x}t}{\varepsilon^2}\right)\right], 0 < \varepsilon < t, \quad (25)$$

converge to the finite-dimensional distributions of the Gaussian vector generalized random field $\mathcal{U}$ *with covariance matrix given by, for* $i, j = 1,\ldots,n$,

$$E[\mathcal{U}_i(t_1,\phi)\mathcal{U}_i(t_2,\varphi)] = \frac{C_1^2 2^{n+2-2\gamma}\mu^2}{C_0^2(t_1t_2)^{n/2+1}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(\mathbf{x}_1) \times \left[\frac{-\partial^2}{\partial\left(x_1^i - x_2^i\right)^2}\|\mathbf{x}_1 - \mathbf{x}_2\|^{-2\gamma}\right]\varphi(\mathbf{x}_2)d\mathbf{x}_1 d\mathbf{x}_2, \quad (26)$$

$$E[\mathcal{U}_i(t_1,\phi)\mathcal{U}_j(t_2,\varphi)] = \frac{C_1^2 2^{n+2-2\gamma}\mu^2}{C_0^2(t_1t_2)^{n/2+1}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(\mathbf{x}_1) \times \left[\frac{-\partial^2}{\partial\left(x_1^j - x_2^j\right)\partial\left(x_1^i - x_2^i\right)}\|\mathbf{x}_1 - \mathbf{x}_2\|^{-2\gamma}\right]\varphi(\mathbf{x}_2)d\mathbf{x}_1 d\mathbf{x}_2, \quad (27)$$

where $0 < \gamma < n/2$ *and* $\phi, \varphi \in H^{\gamma - n/2 + 1/n}(\mathbb{R}^n)$, *with*

$$\mathbf{F}(t,\mathbf{x}) = (F_1(t,\mathbf{x}),\ldots,F_n(t,\mathbf{x})) = \left(2(\mu b)^{1/2}\tanh(\omega t)x_1,\ldots,2(\mu b)^{1/2}\tanh(\omega t)x_n\right) = (\omega\tanh(\omega t)x_1,\ldots,\omega\tanh(\omega t)x_n). \quad (28)$$

Proof: For $i = 1, \ldots, n$, the random components U_i^ε of $\mathbf{U}^\varepsilon$ are defined as

$$U_i^\varepsilon(t, \mathbf{x}) = \frac{-2\mu}{\varepsilon^\gamma} \frac{\varphi(\gamma)^{1/2}}{(2\pi)^{\gamma/2}}$$

$$\left[\frac{\int_{\mathbb{R}^n} \frac{i\lambda_i (2t)^{n/2+1}}{(t^2+\varepsilon^2)^{n/2+1}} \exp\left\{i\left\langle\lambda, \frac{2t^2\mathbf{x}}{t^2+\varepsilon^2}\right\rangle\right\} \exp\left\{-\frac{1}{2}\left(\frac{\mu}{b}\right)^{1/2}\left[\frac{t^2-\varepsilon^2}{t^2+\varepsilon^2}\right]\|\lambda\|^2\right\} Z(d\lambda)}{\int_{\mathbb{R}^n} \exp\left\{i\left\langle\lambda, \frac{2t^2\mathbf{x}}{t^2+\varepsilon^2}\right\rangle\right\} \exp\left\{-\frac{1}{2}\left(\frac{\mu}{b}\right)^{1/2}\left[\frac{t^2-\varepsilon^2}{t^2+\varepsilon^2}\right]\|\lambda\|^2\right\} \frac{Z(d\lambda)}{\left[\frac{t^2+\varepsilon^2}{2t\varepsilon}\right]^{n/2}} + C_0}\right]$$

$$= \frac{1}{\varepsilon^{n/2+2+\gamma}} \frac{\varphi(\gamma)^{1/2}}{(2\pi)^{\gamma/2}} \frac{I_i\left(\frac{1}{\omega}\ln\left[\frac{t}{\varepsilon}\right], \frac{\mathbf{x}t}{\varepsilon^2}\right)}{J\left(\frac{1}{\omega}\ln\left[\frac{t}{\varepsilon}\right], \frac{\mathbf{x}t}{\varepsilon^2}\right)} \tag{29}$$

We first calculate the limit in probability of $J\left(\frac{1}{\omega}\ln\left[\frac{t}{\varepsilon}\right], \frac{\mathbf{x}t}{\varepsilon^2}\right)$. Considering the change of variable $\widetilde{\lambda} = \frac{\lambda}{\varepsilon}$ in the expression defining random field J, we have

$$J\left(\frac{1}{\omega}\ln\left[\frac{t}{\varepsilon}\right], \frac{\mathbf{x}t}{\varepsilon^2}\right) = \int_{\mathbb{R}^n} \exp\left\{i\left\langle\widetilde{\lambda}, \frac{2t^2\mathbf{x}}{t^2+\varepsilon^2}\right\rangle\right\}$$
$$\exp\left\{-\frac{1}{2}\left(\frac{\mu}{b}\right)^{1/2} \frac{(t^2-\varepsilon^2)\varepsilon^2\|\widetilde{\lambda}\|^2}{t^2+\varepsilon^2}\right\} \frac{Z(d\varepsilon\widetilde{\lambda})}{\left[\frac{t^2+\varepsilon^2}{2t\varepsilon}\right]^{n/2}} + C_0, \tag{30}$$

and then, we obtain

$$E\left[J\left(\frac{1}{\omega}\ln\left[\frac{t}{\varepsilon}\right], \frac{\mathbf{x}t}{\varepsilon^2}\right) - C_0\right]^2 = \varepsilon^{2n}\int_{\mathbb{R}}\left[\frac{2t}{t^2+\varepsilon^2}\right]^n$$
$$\times \exp\left\{-\left(\frac{\mu}{b}\right)^{1/2}\left[\frac{t^2-\varepsilon^2}{t^2+\varepsilon^2}\right]\|\widetilde{\lambda}\varepsilon\|^2\right\}\left[\int_{\mathbb{R}} \exp\{-i\langle\widetilde{\lambda}\varepsilon, \mathbf{x}\rangle\}\sum_{k=1}^{\infty}\frac{C_k^2}{k!}B_\xi^k(\mathbf{x})d\mathbf{x}\right]d\widetilde{\lambda}. \tag{31}$$

Since

$$\varepsilon^n\int_{\mathbb{R}}\exp\{-i\langle\widetilde{\lambda}\varepsilon, \mathbf{x}\rangle\}\sum_{k=1}^{\infty}\frac{C_k^2}{k!}B_\xi^k(\mathbf{x})d\mathbf{x} \leq \varepsilon^{2\gamma}\frac{(2\pi)^\gamma}{\varphi(\gamma)}\|\widetilde{\lambda}\|^{-n+2\gamma}(1+\|\widetilde{\lambda}\varepsilon\|^2)^{-\alpha}\sum_{k=1}^{\infty}\frac{C_k^2}{k!}$$

(see Ref. 13), equation (31) converges to zero when ε goes to zero, and the convergence in probability to the constant C_0 follows.

We now study the weak-sense limit of

$$E\left[\frac{1}{\varepsilon^{n/2+2+\gamma}}\frac{\varphi(\gamma)^{1/2}}{(2\pi)^{\gamma/2}}I_i\left(\frac{1}{\omega}\ln\left[\frac{t_1}{\varepsilon}\right], \frac{\mathbf{x}_1 t_1}{\varepsilon^2}\right)\frac{1}{\varepsilon^{n/2+2+\gamma}}\frac{\varphi(\gamma)^{1/2}}{(2\pi)^{\gamma/2}}I_i\left(\frac{1}{\omega}\ln\left[\frac{t_2}{\varepsilon}\right], \frac{\mathbf{x}_2 t_2}{\varepsilon^2}\right)\right]$$
$$= 4\mu^2\frac{\varphi(\gamma)}{\varepsilon^{2\gamma}(2\pi)^\gamma}\int_{\mathbb{R}^n}\widetilde{\lambda}_i^2\frac{2^{n+2}(t_1t_2)^{n/2+1}\varepsilon^n}{([t_1^2+\varepsilon^2][t_2^2+\varepsilon^2])^{n/2+1}}\exp\left\{i\left\langle\widetilde{\lambda}, \left[\frac{2\mathbf{x}_1 t_1^2}{t_1^2+\varepsilon^2} - \frac{2\mathbf{x}_2 t_2^2}{t_2^2+\varepsilon^2}\right]\right\rangle\right\}$$

$$\times \exp\left\{-\frac{\|\widetilde{\lambda}\|^2}{2}\left(\frac{\mu}{b}\right)^{1/2}\varepsilon^2\left[\frac{t_1^2-\varepsilon^2}{t_1^2+\varepsilon^2}+\frac{t_2^2-\varepsilon^2}{t_2^2+\varepsilon^2}\right]\right\}$$

$$\times \int_{\mathbb{R}^n} \exp\left\{-i\left\langle \varepsilon\widetilde{\lambda}, \mathbf{x}\right\rangle\right\} \sum_{k=2}^{\infty} \frac{C_k^2}{k!} B_\xi^k(\mathbf{x}) d\mathbf{x} d\widetilde{\lambda}$$

$$+4\mu^2 \frac{\varphi(\gamma)}{\varepsilon^{2\gamma}(2\pi)^\gamma} \int_{\mathbb{R}^n} \widetilde{\lambda}_i^2 \frac{2^{n+2}(t_1t_2)^{n/2+1}\varepsilon^n}{([t_1^2+\varepsilon^2][t_2^2+\varepsilon^2])^{n/2+1}} \exp\left\{i\left\langle\widetilde{\lambda}, \left[\frac{2\mathbf{x}_1 t_1^2}{t_1^2+\varepsilon^2} - \frac{2\mathbf{x}_2 t_2^2}{t_2^2+\varepsilon^2}\right]\right\rangle\right\}$$

$$\times \exp\left\{-\frac{\|\widetilde{\lambda}\|^2}{2}\left(\frac{\mu}{b}\right)^{1/2}\varepsilon^2\left[\frac{t_1^2-\varepsilon^2}{t_1^2+\varepsilon^2}+\frac{t_2^2-\varepsilon^2}{t_2^2+\varepsilon^2}\right]\right\}$$

$$\times \int_{\mathbb{R}^n} \exp\left\{-i\left\langle \varepsilon\widetilde{\lambda}, \mathbf{x}\right\rangle\right\} C_1^2 B_\xi(\mathbf{x}) d\mathbf{x} d\widetilde{\lambda} = A_i^1 + A_i^2, \tag{32}$$

where we have previously considered in the expression defining $I_i(\frac{1}{\omega}\ln[\frac{t_1}{\varepsilon}], \frac{\mathbf{x}_1 t_1}{\varepsilon^2})$ the change of variables $\widetilde{\lambda} = \frac{\lambda}{\varepsilon}$. The quantities A_i^1 and A_i^2 respectively represent the covariance functions of two random fields V_i^1 and V_i^2 for $i = 1, \ldots, n$.

We first study the limit in probability of V_i^1. Since, for ε sufficiently small,

$$\varepsilon^n \left[\int_{\mathbb{R}^n} \exp\{-i\langle\varepsilon\widetilde{\lambda}, \mathbf{x}\rangle\} \sum_{k=2}^{\infty} \frac{C_k^2}{k!} B_\xi^k(\mathbf{x}) d\mathbf{x}\right] \le \varepsilon^n \left[\int_{\mathbb{R}^n} \exp\{-i\langle\varepsilon\widetilde{\lambda}, \mathbf{x}\rangle\} B_\xi^2(\mathbf{x}) d\mathbf{x}\right] \sum_{k=2}^{\infty} \frac{C_k^2}{k!},$$

and

$$\varepsilon^n \int_{\mathbb{R}^n} \exp\left\{-i\left\langle\varepsilon\widetilde{\lambda}, \mathbf{x}\right\rangle\right\} B_\xi^2(\mathbf{x}) d\mathbf{x}$$

goes to zero like function $\varepsilon^{4\gamma}$, we then have that $\mathrm{Var}(V_i^1)$ goes to zero like function $\varepsilon^{2\gamma}$, when ε goes to zero.

We now study the limiting covariance of V_i^2. From equation (32),

$$\begin{aligned}\lim_{\varepsilon\to 0} A_i^2 &= \frac{C_1^2 2^{n+4}\mu^2}{(t_1t_2)^{n/2+1}} \int_{\mathbb{R}^n} \widetilde{\lambda}_i^2 \exp\left\{i\left\langle\widetilde{\lambda}, 2(\mathbf{x}_1-\mathbf{x}_2)\right\rangle\right\} \|\widetilde{\lambda}\|^{-n+2\gamma} d\widetilde{\lambda} \\ &= \frac{C_1^2 2^{n+2-2\gamma}\mu^2}{(t_1t_2)^{n/2+1}} - \frac{\partial^2}{\partial\left(x_1^i - x_2^i\right)^2} \|\mathbf{x}_1 - \mathbf{x}_2\|^{-2\gamma},\end{aligned} \tag{33}$$

which defines the covariance function of a generalized random field $\mathcal{U}_i$ on $H^{\gamma-n/2+1/n}(\mathbb{R}^n)$ given by

$$\mathcal{U}_i(t,\varphi) = \frac{2^{\frac{n+2}{2}} C_{1\mu}}{2^\gamma t^{n/2+1}} \int_{\mathbb{R}^n} \varphi(\mathbf{x}) \int_{\mathbb{R}^n} \exp\{i\langle\omega, \mathbf{x}\rangle\} \|\omega\|^{-\frac{n}{2}+\gamma} i\omega_i \widehat{\varepsilon}(d\omega) d\mathbf{x}. \tag{34}$$

The limit given by equation (27) is obtained in a similar way to equation (26).

From the limits calculated, the application of Slutskii lemma[3] leads to the desired result on convergence of finite-dimensional distributions of each component of the vector generalized random field $\mathcal{U}_\varepsilon$ to the finite-dimensional distributions of the corresponding component of the vector generalized random field $\mathcal{U}$. □

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[3] Slutsky's Lemma was originally proved by Slutsky, E(1925), Über stochastische asymptoten and grenzwerte. Metron, 5, 1-90. The proof can be found also in most book on foundation of mathematical statistics, for instance on pp 346-348 of the book: Nguyen, H.T. and Rogers, G.S. (1989), Foundation of Mathematical Statistics, Springer-Verlag. We need a multidimensional version of this lemma, see for instance, Leonenko,(24) p. 227: In the sequel, we shall use the notations $\overset{P}{\to}$ and $\overset{D}{\to}$ to denote convergence in probability and the convergence in distributions respectively.

Lemma 4.3.1. Let $\{u_t\}$ and $\{v_t\}$ are families of random vectors from R^n and $\{w_t\}$ is family of random variables and let $u_t \overset{D}{\to} u, v_t \overset{P}{\to} c = (c_1, \dots, c_n)$ as $t \to \infty$, where $c_i = const,\ i = \overline{i, n}$ and $w_t \overset{P}{\to} d = const$. Consequently as $t \to \infty : u_t + v_t \overset{D}{\to} u + c, w_t u_t \overset{D}{\to} du$ and if $d \neq 0$, then $u_t/w_t \overset{D}{\to} u/d$.

In our paper we use this statement with $\varepsilon = 1/t$, and also we need an adaptation of this result to the convergence in distribution of the generalized random fieds, this can be done similar to the proof of the Theorem 3.1 in the paper: Surgailis and Woyczynski(14) Long-range prediction and scaling limit for statistical solutions of the Burgers' equation, in Nonlinear Waves and Weak Turbulence (Fitzmaurice, N., Gurarie, D., F.McCaughen and W.A. Woyczynski, eds), Birkhäuser, Boston, pp. 313–338.

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